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Algebraic methods to compute Mathieu functions

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Abstract

The standard form of the Mathieu differential equation is $y''(z) + (a - 2q\cos 2z)\,y(z) = 0$, where a is the characteristic number and q is a real parameter. The most useful solution forms are given in terms of expansions for either small or large values of q. In this paper we obtain closed formulae for the generic term of expansions of Mathieu functions in the following cases:

(1) standard series expansion for small q;
(2) Fourier series expansion for small q;
(3) asymptotic expansion in terms of trigonometric functions for large q and
(4) asymptotic expansion in terms of parabolic cylinder functions for large q.

We also obtain closed formulae for the generic term of expansions of characteristic numbers and normalization formulae for small and large q. Using these formulae one can efficiently generate high-order expansions that can be used for implementation of the algebraic aspects of Mathieu functions in computer algebra systems. These formulae also provide alternative methods for numerical evaluation of Mathieu functions.

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1. Introduction

Mathieu functions were introduced by Mathieu [1] when analysing the movements of membranes of elliptical shape. Since then, many physical and astronomical problems have been reported as requiring these functions in their analysis [2]. What attracts our attention is that the number of applications of Mathieu functions is much smaller then the applications of other functions of mathematical physics, such as Bessel or hypergeometric functions. What can explain this state of affairs? A possible answer can be found in the work of Whittaker [3], 'their actual analytical determination presents great difficulties', or in that of Sips [4], 'les fonctions de Mathieu restent d'un emploi difficile, principalement par suite de l'impossibilité de les reprèsenter analytiquement d'une manière simple et maniable'[1].

[1] Mathieu functions remain difficult to employ, mainly because of the impossibility of analytically representing them in a simple and handy way.

One purpose of this paper is to contribute to decreasing the lack of analytical expressions by providing closed-form solutions for the generic term of expansions of Mathieu functions and characteristic numbers. We consider two forms of expansion for small values of parameter q (see equation (1) below) and two forms of asymptotic expansion for large q. In all cases the generic terms are given by recursive equations that were obtained after laborious and lengthy algebraic manipulations, but they can be easily implemented in computer algebra systems. An experimental implementation performed by us was used to test these equations and it shows that the results published in the literature can be quickly reproduced and improved.

This paper can be used for a robust implementation of Mathieu functions and characteristic numbers in computer algebra systems, since high-order expansions for small and large values of q are desirable features in the implementation of their algebraic aspects. A second application is to provide an alternative method for their numerical evaluation. In general, the use of series expansions is the most efficient method for numerical evaluation of functions. This method can only be employed if the generic term of the series expansion is known, since increasing numerical precision requires the calculation of higher terms in the series expansions.

This paper deals with functions of the first kind. In section 2, closed-form expansions for small values of parameter q are presented in the standard form introduced by Mathieu and the Fourier series. In sections 3 and 4, two different asymptotic expansions are considered: the first given in terms of trigonometric functions, and the second in terms of parabolic cylinder functions. In section 5, closed formulae for the generic term of the normalization formulae for small and large values of parameter q are presented.

2. Expansions for small q

The original form of Mathieu's differential equation is [1]

$$\frac{\mathrm{d}^2 y}{\mathrm{d}z^2} + (a - 2\,q\cos 2z)\,y = 0 \tag{1}$$

where a and q are real parameters and z can be complex. The solutions for $q = 0$ and $a = r^2$ are $y_1 = \cos(rz)$ and $y_2 = \sin(rz)$. Let us look for solutions of (1) that have the forms

$$ce_r(z, q) = \cos(rz) + \sum_{k=1}^{\infty} c_k^{(r)}(z)\,q^k \tag{2a}$$

$$se_r(z, q) = \sin(rz) + \sum_{k=1}^{\infty} s_k^{(r)}(z)\,q^k \tag{2b}$$

since they reduce to y_1 and y_2 when $q = 0$. Functions $ce_r(z, q)$ and $se_r(z, q)$ are called *cosine* and *sine elliptic* respectively. We omit the index r of the coefficients $c_k(z)$ and $s_k(z)$ and take $c_0 = \cos(rz)$ and $s_0 = \sin(rz)$ in order to simplify the notation. In this paper we assume that r is a positive real parameter unless stated otherwise. The rth characteristic values a (for cosine) and b (for sine) can be expressed as

$$a_r = r^2 + \sum_{k=1}^{\infty} \alpha_k^{(r)}\,q^k \tag{3a}$$

$$b_r = r^2 + \sum_{k=1}^{\infty} \beta_k^{(r)}\,q^k \tag{3b}$$

which also reduce to the desired formula when $q = 0$. The coefficients $\alpha_k^{(r)}$ and $\beta_k^{(r)}$ are obtained by imposing that the functions $ce_r(z, q)$ and $se_r(z, q)$ are periodic in z. Mathieu functions are solutions with period π and 2π, and, in this case, r must be an integer. If r is not an integer, equations (2) are still valid solutions of (1).

Substituting (2) and (3) into equation (1), we obtain the differential equations for the coefficients c_k and s_k

$$\frac{\mathrm{d}^2 c_k}{\mathrm{d}z^2} + r^2 c_k - 2\cos(2z)c_{k-1} + \sum_{l=0}^{k-1} \alpha_{k-l} c_l = 0 \tag{4a}$$

$$\frac{\mathrm{d}^2 s_k}{\mathrm{d}z^2} + r^2 s_k - 2\cos(2z)s_{k-1} + \sum_{l=0}^{k-1} \beta_{k-l} s_l = 0 \tag{4b}$$

valid for $k \geqslant 1$. With no loss of generality, we adopt the following ansatz for the solutions of (4a):

$$c_k = \sum_{i=-\lfloor \frac{k}{2} \rfloor}^{\lfloor \frac{k+1}{2} \rfloor} A^k_{4i-2k+4\lfloor \frac{k}{2} \rfloor} \cos\left(r + 4i - 2k + 4\left\lfloor \frac{k}{2} \right\rfloor\right) z \tag{5}$$

where $\lfloor \frac{k}{2} \rfloor$ is the greatest integer less or equal to $\frac{k}{2}$. Substituting (5) into equation (4a), factoring terms with the cosine of the same argument and imposing that the coefficients vanish, we obtain the following recursive equations:

$$A^k_{2i} = \frac{1}{4i(r+i)} \left(-A^{k-1}_{2i+2} - A^{k-1}_{2i-2} + \sum_{j=1}^{\frac{k+1-|r+i|}{2}} \alpha_{2j-1} A^{k-2j+1}_{-2i-2r} + \sum_{j=1}^{\frac{k-|i|}{2}} \alpha_{2j} A^{k-2j}_{2i} \right) \qquad r \text{ odd} \tag{6a}$$

$$A^k_{2i} = \frac{1}{4i(r+i)} \left(-A^{k-1}_{2i+2} - A^{k-1}_{2i-2} + \sum_{j=1}^{\frac{k-|i|}{2}} \alpha_{2j} A^{k-2j}_{2i} \right) \qquad \text{otherwise} \tag{6b}$$

where $|i| \leqslant k$ and $k \geqslant 1$. If the upper limit of a sum is less than unity, it is replaced by zero. We take $A^k_{2i} = 0$ if $|i| > k$, since these coefficients are outside the index range of equation (5). A^0_0 is a non-determined constant which can be replaced by unity. By taking $i = k$ and $-k$ in equation (6), we obtain non-recursive formulae for these special cases.

$$A^k_{2k} = \frac{(-1)^k}{4^k k!(r+1)_k}, \tag{7}$$

$$A^k_{-2k} = \begin{cases} \dfrac{(-1)^k}{4^k k!(1-r)_k} & k < r \quad \text{if} \quad r \text{ integer} \quad (8) \\ 0 & r \text{ even} \quad \text{and} \quad k > r \quad (9) \end{cases}$$

where $(r)_k$ is the Pochhammer symbol. A^k_0 and A^k_{-2r} cannot be obtained from equations (6), since the denominator vanishes at $i = 0$ and $-r$. They are coefficients of $\cos(rz)$ and can be chosen arbitrarily, since they can be eliminated by some multiplicative constant. In order to obtain McLachlan's results [2], we have to use the conventions $A^k_0 = 0$ and $A^k_{-2r} = 0$, while the results of Abramowitz and Stegun [5] are obtained with non-null values.

In order to find the values of α_k and β_k that make the solutions of the Mathieu equation periodic, we start by noticing that equations (4) are recursive differential equations. Using equation (5) we can eliminate the recursion, obtaining ordinary differential equations for c_k and s_k, which can be fully integrated. By imposing that the coefficients of the non-periodic part are zero we obtain

$$\alpha_{2j-1} = \begin{cases} A^{2j-2}_{2-2r} + A^{2j-2}_{-2-2r} & r \text{ odd} \\ 0 & \text{otherwise} \end{cases} \tag{10a}$$

$$\alpha_{2j} = \begin{cases} A^{2j-1}_{2} + A^{2j-1}_{-2} + A^{2j-1}_{2-2r} + A^{2j-1}_{-2-2r} & r \text{ even} \quad \text{and} \quad r \neq 0 \\ A^{2j-1}_{2} + A^{2j-1}_{-2} & \text{otherwise.} \end{cases} \tag{10b}$$

For example, replacing r by unity in equation (3a) and calculating $\alpha_k^{(1)}$ using equation (10a) when k is odd and equation (10b) when k is even, in both cases using (6a) with $r = 1$ and $A_0^0 = 1$, we can obtain the following expansion for a_1 up to order $O(q^{13})$:

$$\begin{aligned} a_1 = {} & 1 + q - \tfrac{1}{8}q^2 - \tfrac{1}{64}q^3 - \tfrac{1}{1536}q^4 + \tfrac{11}{36\,864}q^5 + \tfrac{49}{589\,824}q^6 + \tfrac{55}{9437\,184}q^7 \\ & - \tfrac{83}{35\,389\,440}q^8 - \tfrac{12\,121}{15\,099\,494\,400}q^9 - \tfrac{114\,299}{1\,630\,745\,395\,200}q^{10} \\ & + \tfrac{192\,151}{7827\,577\,896\,960}q^{11} + \tfrac{83\,513\,957}{8766\,887\,244\,595\,200}q^{12} + O\left(q^{13}\right). \end{aligned} \tag{11}$$

The example above and the following ones are calculated using a code implemented in Maple[2]. If r is not an integer, the characteristic numbers are given by only one expression, which is

$$\begin{aligned} a_r = {} & r^2 + \frac{1}{2}\,\frac{1}{r^2-1}q^2 + \frac{1}{32}\,\frac{5\,r^2+7}{\left(r^2-4\right)\left(r^2-1\right)^3}q^4 + \frac{1}{64}\,\frac{9\,r^4+58\,r^2+29}{\left(r^2-4\right)\left(r^2-9\right)\left(r^2-1\right)^5}q^6 \\ & + \frac{1469\,r^{10}+9144\,r^8-140\,354\,r^6+64\,228\,r^4+827\,565\,r^2+274\,748}{8192\left(r^2-9\right)\left(r^2-16\right)\left(r^2-4\right)^3\left(r^2-1\right)^7}q^8 \\ & + O\left(q^{10}\right) \end{aligned} \tag{12}$$

up to order $O(q^{10})$, which agrees with formula 20.2.26 of [5]. The expansion above yields the correct expression up to order $O(q^{10})$ for integer r if $r \geqslant 10$.

We take the following ansatz for the solutions of (4b):

$$s_k = \sum_{i=-\lfloor \frac{k}{2} \rfloor}^{\lfloor \frac{k+1}{2} \rfloor} B^k_{4i-2k+4\lfloor \frac{k}{2} \rfloor} \sin\left(r + 4i - 2k + 4\left\lfloor \frac{k}{2} \right\rfloor\right) z. \tag{13}$$

In this case, the recursive equations for the coefficients B^k_{2i} are

$$B^k_{2i} = \frac{1}{4i\left(r+i\right)}\left(-B^{k-1}_{2i+2} - B^{k-1}_{2i-2} - \sum_{j=1}^{\frac{k+1-|r+i|}{2}} \beta_{2j-1} B^{k-2j+1}_{-2i-2r} + \sum_{j=1}^{\frac{k-|i|}{2}} \beta_{2j} B^{k-2j}_{2i}\right) \qquad r \text{ odd} \tag{14a}$$

$$B^k_{2i} = \frac{1}{4i\left(r+i\right)}\left(-B^{k-1}_{2i+2} - B^{k-1}_{2i-2} + \sum_{j=1}^{\frac{k-|i|}{2}} \beta_{2j} B^{k-2j}_{2i}\right) \qquad \text{otherwise} \tag{14b}$$

where $B^k_{2i} = 0$ if $|i| > k$, $B^k_0 = B^k_{-2r} = 0$ and $B^0_0 = 1$. For $i = k$ and $-k$, B^k_{2i} are equal to A^k_{2i}, which are given by equations (7)–(8). The values of β that make the solutions periodic are

$$\beta_{2j-1} = \begin{cases} -B^{2j-2}_{2-2r} - B^{2j-2}_{-2-2r} & r \text{ odd} \\ 0 & \text{otherwise} \end{cases} \tag{15a}$$

$$\beta_{2j} = \begin{cases} B^{2j-1}_{2} + B^{2j-1}_{-2} - B^{2j-1}_{2-2r} - B^{2j-1}_{-2-2r} & r \text{ even} \\ B^{2j-1}_{2} + B^{2j-1}_{-2} & \text{otherwise.} \end{cases} \tag{15b}$$

[2] Maple Waterloo Software, Inc. See http://www.maplesoft.com

These recursive formulae have already been calculated in a different form by Rubin [6].

For integer r, the preferred form [2] of the solutions of Mathieu equation (1) is the Fourier expansions

$$ce_r(z,q) = \frac{De_0^{(r)}}{2} + \sum_{m=1}^{\infty} De_m^{(r)} \cos(mz) \tag{16a}$$

$$se_r(z,q) = \sum_{m=0}^{\infty} Do_m^{(r)} \sin(mz) \tag{16b}$$

where the coefficients $De_m^{(r)}$ and $Do_m^{(r)}$ are

$$De_m^{(r)} = \delta_{r,m}\,(1+\delta_{r,0}) + \sum_{k=1}^{\infty}(A_{m-r}^k + A_{-m-r}^k)\,q^k \tag{17a}$$

$$Do_m^{(r)} = \delta_{r,m} + \sum_{k=1}^{\infty}(B_{m-r}^k - B_{-m-r}^k)\,q^k. \tag{17b}$$

For non-integer r, the expansions are given by

$$ce_r = \sum_{m=-\infty}^{\infty} De_{2m}^{(r)} \cos\big(r+2m\big)z \tag{18a}$$

$$se_r = \sum_{m=-\infty}^{\infty} Do_{2m}^{(r)} \sin\big(r+2m\big)z \tag{18b}$$

and the coefficients $De_{2m}^{(r)}$ and $Do_{2m}^{(r)}$ are

$$De_{2m}^{(r)} = \sum_{k=\left\lfloor\frac{|m|+1}{2}\right\rfloor}^{\infty} q^{2k-|m|+2\left\lfloor\frac{|m|}{2}\right\rfloor} A_{2m}^{2k-|m|+2\left\lfloor\frac{|m|}{2}\right\rfloor} \tag{19a}$$

$$Do_{2m}^{(r)} = \sum_{k=\left\lfloor\frac{|m|+1}{2}\right\rfloor}^{\infty} q^{2k-|m|+2\left\lfloor\frac{|m|}{2}\right\rfloor} B_{2m}^{2k-|m|+2\left\lfloor\frac{|m|}{2}\right\rfloor}. \tag{19b}$$

It is interesting to verify whether the formulae of this section can be used for efficient numerical evaluation of Mathieu functions and characteristic numbers. The great majority of numerical methods requires an estimated value to start the computation [7], since in general the series have finite convergence radii and Newton-like methods must be used. In principle, the formulae of this section can be used to improve the precision of a starting value. Due to the limited size of convergence radii of the series for small q, it is important to find formulae for the generic coefficient of asymptotic series (large q) as well. This is addressed in the following sections.

3. Asymptotic expansion

An ansatz for the asymptotic expansion of $ce_r(z,q)$ in the interval $-\frac{\pi}{2} < z < \frac{\pi}{2}$ for integer r is [2]

$$ce_r(z,q) = \mathrm{e}^{\chi^+(z)}\zeta^-(z)\sum_{k=0}^{\infty}\frac{f_k^+(z)}{q^{\frac{k}{2}}} + \mathrm{e}^{\chi^-(z)}\zeta^+(z)\sum_{k=0}^{\infty}\frac{f_k^-(z)}{q^{\frac{k}{2}}} \tag{20}$$

where

$$\chi^{\pm}(z) = \pm 2\sqrt{q}\sin(z) \tag{21}$$

$$\zeta^{\pm}(z) = \frac{(\sec(z)+\tan(z))^{\pm(r+1/2)}}{\sqrt{\cos(z)}} \tag{22}$$

$$f_k^{\pm}(z) = \sum_{i=0}^{k} \frac{F_{2i}^k \pm E_{2i}^k \sin(z)}{\cos(z)^{2i}}. \tag{23}$$

The characteristic numbers have the form [2]

$$b_{r+1} \simeq a_r = -2\,q + \left(4\,r+2\right)\sqrt{q} - \frac{1}{2}\left(r^2 + r + \frac{1}{2}\right) + \sum_{k=1}^{\infty} \frac{\gamma_k^{(r)}}{q^{\frac{k}{2}}}. \tag{24}$$

Substituting equations (20) and (24) into Mathieu equation (1), factoring like terms and imposing that the coefficients vanish, we obtain the following recursive formulae for E_{2i}^k and F_{2i}^k:

$$E_{2i}^k = \frac{1}{4\,(2\,i-1)}\Bigg(8\,(i-1)\,E_{2i-2}^k - (2\,i-3)\,(1+2\,r)\,E_{2i-4}^{k-1} + 2\,(i-1)\,(1+2\,r)\,E_{2i-2}^{k-1} + ((2\,i-3)^2 + r + r^2)\,F_{2i-4}^{k-1} - \frac{(4\,i-3)^2}{4}\,F_{2i-2}^{k-1} + \sum_{j=1}^{k-i+1} \gamma_{k-j-i+1}^{(r)}\,F_{2i-2}^{i+j-2}\Bigg) \tag{25}$$

and

$$F_{2i}^k = \frac{1}{8i}\Bigg(\frac{(4\,i-1)^2}{4}\,E_{2i}^{k-1} - ((2\,i-1)^2 + r + r^2)\,E_{2i-2}^{k-1} + (2\,i-1)\,(1+2\,r)\,F_{2i-2}^{k-1} - \sum_{j=1}^{k-i} \gamma_{k-j-i}^{(r)}\,E_{2i}^{i+j-1}\Bigg) \tag{26}$$

where $E_0^0 = 0$, $F_0^0 = 1$ and $E_{2i}^k = F_{2i}^k = 0$ if $k = 0$ or $i < 0$. The recursive equation for $\gamma_k^{(r)}$ is obtained by imposing that the coefficients of the non-periodic terms vanish

$$\gamma_k^{(r)} = \frac{1}{2\sqrt{\pi}} \sum_{i=1}^{k} \frac{\Gamma(i+\frac{1}{2})}{\Gamma(i+1)}\Bigg(\frac{(1+2\,r)}{i+1}\,E_{2i}^k - \frac{1}{2}\,\frac{(2\,r\,(2\,i+1)\,(r+1)+1+3\,i)}{i+1}\,F_{2i}^k - 2\sum_{j=1}^{k-i+1} \gamma_{k-j-i+1}^{(r)}\,F_{2i}^{i+j-1}\Bigg). \tag{27}$$

The expression for $ce_r(z, q)$ given in (20) is an even function, since $\zeta^-(-z) = \zeta^+(z)$ and $f_k^+(-z) = f_k^-(z)$. The corresponding asymptotic expansion for the function $se_r(z, q)$ in the interval $\frac{\pi}{2} < z < \frac{3\pi}{2}$ is the odd function

$$se_{r+1}(z, q) = \mathrm{e}^{\chi^+(z)}\zeta^-(z)\sum_{k=0}^{\infty} \frac{f_k^+(z)}{q^{\frac{k}{2}}} - \mathrm{e}^{\chi^-(z)}\zeta^+(z)\sum_{k=0}^{\infty} \frac{f_k^-(z)}{q^{\frac{k}{2}}}. \tag{28}$$

Note that the asymptotic expansion has the same form for all allowed values of r, while the expansions around $q = 0$ have different forms depending on r being even, odd or non-integer.

We show some examples using the formulae above. The asymptotic expansion of $ce_r(z, q)$ up to order $\mathrm{O}(\frac{1}{q^{3/2}})$ is

$$ce_r(z, q) = \mathrm{e}^{2\sqrt{q}\sin(z)}\,\frac{\left(\sec(z)+\tan(z)\right)^{-r-1/2}}{\sqrt{\cos(z)}}\left(1 - \frac{\left(r^2+r+1\right)\sin(z) - 2\,r - 1}{8\sqrt{q}\cos(z)^2}\right.$$

$$+\frac{1}{128\,q\cos(z)^4}\big[(-(2\,r+1)(r^2+r+1)\cos(z)^2$$
$$-4\,(2\,r+1)(r+3+r^2))\sin(z)-(r^2+r+5)(r^2+r+1)\cos(z)^2$$
$$+12+23\,r^2+22\,r+2\,r^3+r^4\big]+\mathrm{O}\left(\frac{1}{q^{3/2}}\right)\bigg)+(z\to -z) \tag{29}$$

and the expansion of a_r up to order $\mathrm{O}(\frac{1}{q^{3/2}})$ is

$$\begin{aligned}a_r &= -2\,q+(2+4\,r)\sqrt{q}-\tfrac{1}{4}-\tfrac{1}{2}\,r-\tfrac{1}{2}\,r^2\\ &\quad+\left(-\frac{1}{32}-\frac{3}{32}\,r-\frac{3}{32}\,r^2-\frac{1}{16}\,r^3\right)\frac{1}{\sqrt{q}}\\ &\quad+\left(-\frac{3}{256}-\frac{11}{256}\,r-\frac{1}{16}\,r^2-\frac{5}{128}\,r^3-\frac{5}{256}\,r^4\right)\frac{1}{q}+\mathrm{O}\left(\frac{1}{q^{3/2}}\right).\end{aligned} \tag{30}$$

Equations (29) and (30) agree with formulae (11.43.5) and (11.44.1) of [2].

Note that these asymptotic expansions for $ce_r(z,q)$ and $se_r(z,q)$ have an essential singularity at $z=\frac{\pi}{2}$. This fact makes it impractical to use these expansions for numerical evaluation for z close to $\frac{\pi}{2}$. In the next section we analyse an alternative asymptotic expansion which gives good results for this range of z. Dingle and Müller [8] point out that two kinds of asymptotic expansion are required to cover the range $[0,2\pi]$.

4. Alternative asymptotic expansion

An alternative ansatz for the asymptotic expansions of $ce_r(z,q)$ and $se_r(z,q)$ for integer r in terms of the parabolic cylinder functions $\mathrm{D}_m(\alpha)$ is [4, 9, 10]

$$ce_r(z,q)=\sum_{k=0}^{\infty}\frac{1}{4^k q^{k/2}}\left(Q_{0,k}^{(r)}+Q_{1,k}^{(r)}\right) \tag{31a}$$

$$se_{r+1}(z,q)=\sin(z)\sum_{k=0}^{\infty}\frac{1}{4^k q^{k/2}}\left(Q_{0,k}^{(r)}-Q_{1,k}^{(r)}\right) \tag{31b}$$

where

$$Q_{p,k}^{(r)}=\sum_{i=p-k}^{k}G_{4i-2p}^{k}\,\mathrm{D}_{r+4i-2p}(\alpha) \tag{32}$$

$$\alpha=2\,q^{1/4}\cos(z) \tag{33}$$

$p=0$ or 1. The functions $\mathrm{D}_m(\alpha)$ satisfy the differential equation

$$\frac{\mathrm{d}^2\mathrm{D}_m(\alpha)}{\mathrm{d}\alpha^2}+\left(m+\frac{1}{2}-\frac{\alpha^2}{4}\right)\mathrm{D}_m(\alpha)=0. \tag{34}$$

They are given by

$$\mathrm{D}_m(\alpha)=(-1)^m\,\mathrm{e}^{\alpha^2/4}\,\frac{\mathrm{d}^m}{\mathrm{d}\alpha^m}(\mathrm{e}^{-\alpha^2/2}) \tag{35}$$

or in terms of Hermite polynomials H_m

$$\mathrm{D}_m(\alpha)=\frac{1}{2^{m/2}}\mathrm{e}^{-\alpha^2/4}H_m\left(\frac{\alpha}{\sqrt{2}}\right). \tag{36}$$

In terms of variable α, Mathieu equation (1) is written as

$$\left(4\sqrt{q}-\alpha^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}\alpha^2}-\alpha\frac{\mathrm{d}y}{\mathrm{d}\alpha}+(a+2\,q-\sqrt{q}\,\alpha^2)\,y=0 \tag{37}$$

where a is given by (24). Substituting equations (24), (31) and (32) into the differential equation (37), factoring like terms and imposing that the coefficients vanish, we obtain the following recursive equation:

$$G_{2i}^{k} = -\frac{1}{4\,i}\left(\frac{1}{2}\,G_{2\,i-4}^{k-1} + G_{2\,i-2}^{k-1} - \left[(r+2\,i)^2 + r + 2\,i + \frac{1}{2}\right]G_{2\,i}^{k-1}\right.$$
$$\left. -(r+2\,i+1)_2\,G_{2\,i+2}^{k-1} + \tfrac{1}{2}\,(r+2\,i+1)_4\,G_{2\,i+4}^{k-1} + \frac{1}{2}\sum_{j=1}^{k-\left\lfloor\frac{|i|+1}{2}\right\rfloor} 4^j\,\gamma_{j-1}^{(r)}\,G_{2\,i}^{k-j}\right) \quad (38)$$

where $k > 0$, $G_0^0 = 1$, $G_{2i}^0 = 0$ and $G_0^k = 0$.

By taking $i = 2k - 1$ and $i = 2k$ in equation (38), we obtain the following non-recursive formula for these limiting cases:

$$G_{2i}^{k} = \frac{(-1)^k}{4^i\left(k - i + 2\left\lfloor\frac{i}{2}\right\rfloor\right)!} \quad (39)$$

and taking $i = -2k$ and $-2k + 1$ we obtain

$$G_{2i}^{k} = \frac{(-4)^i}{\left(k - i + 2\left\lfloor\frac{i}{2}\right\rfloor\right)!}\sum_{m=0}^{-2\,i} S_{-2\,i,\,m}\,r^m \quad (40)$$

where $S_{k,\,m}$ are the Stirling numbers of the first kind.

The expression for $\gamma_k^{(r)}$ for $k \geqslant 1$ (see equation (24)) in terms of G_{2i}^k is obtained by imposing that the coefficients of the non-periodic terms vanish:

$$\gamma_k^{(r)} = -\frac{1}{4^{(k+1)}}\left(G_{-4}^k + 2\,G_{-2}^k - 2\,(r+1)_2\,G_2^k + (r+1)_4\,G_4^k\right). \quad (41)$$

For example, the asymptotic expansion for $ce_r(z, q)$ up to order $\mathrm{O}(\frac{1}{q^{3/2}})$, calculated with the formulae above, is

$$ce_r(z,q) = \mathrm{D}_r + \frac{\left(r^4 - 6\,r^3 + 11\,r^2 - 6\,r\right)\mathrm{D}_{r-4} + \left(4\,r - 4\,r^2\right)\mathrm{D}_{r-2} - 4\,\mathrm{D}_{r+2} - \mathrm{D}_{r+4}}{64\,\sqrt{q}}$$
$$+\frac{1}{1024\,q}\Big[\tfrac{1}{8}\left(r^8 - 28\,r^7 + 322\,r^6 - 1960\,r^5 + 6769\,r^4 - 13\,132\,r^3\right.$$
$$\left.+13\,068\,r^2 - 5040\,r\right)\mathrm{D}_{r-8}$$
$$-\left(r^6 - 15\,r^5 + 85\,r^4 - 225\,r^3 + 274\,r^2 - 120\,r\right)\mathrm{D}_{r-6}$$
$$+4\left(r^5 - 7\,r^4 + 17\,r^3 - 17\,r^2 + 6\,r\right)\mathrm{D}_{r-4} - \left(r^4 + 26\,r^3 - 37\,r^2 + 10\,r\right)\mathrm{D}_{r-2}$$
$$+\left(-36 - 25\,r + r^2\right)\mathrm{D}_{r+2} - 4\left(r+2\right)\mathrm{D}_{r+4} + \mathrm{D}_{r+6} + \tfrac{1}{8}\,\mathrm{D}_{r+8}\Big] + \mathrm{O}\left(\frac{1}{q^{3/2}}\right) \quad (42)$$

which agrees with formulae (3.10) and (3.11) of [10].

5. Normalization

The normalization formulae for $ce_m(z, q)$ and $se_m(z, 0)$ for positive integer values of m and n are

$$\int_0^{2\pi} ce_m(z,q)\,ce_n(z,q)\,\mathrm{d}z = \pi\,\delta_{m,\,n} \quad (43a)$$

$$\int_0^{2\pi} se_m(z,q)\,se_n(z,q)\,\mathrm{d}z = \pi\,\delta_{m,\,n} \quad (43b)$$

$$\int_0^{2\pi} ce_m(z,q)\, se_n(z,q)\,\mathrm{d}z = 0. \tag{43c}$$

For $q = 0$ and $m = 0$ the normalization value is 2π instead of π since $ce_0(z,0) = 1$. In this section, we determine the multiplicative constants C_r and S_r such that equations (2) in the form of a Taylor series in q and equations (31) in the form of an asymptotic expansion in q obey equations (43).

Multiplying $ce_r(z,q)$ and $se_r(z,q)$, given by equations (2a) and (2b), by C_r and S_r respectively, substituting them into equations (43) and using the orthogonality of sine and cosine functions, one eventually obtains the following expressions for the normalizing factors for integer r:

$$\frac{1}{C_r^2} = 1 + \frac{1}{1+\delta_{r,0}} \sum_{k=1}^{\infty}\sum_{l=1}^{k-1} \sum_{i=r-2\lfloor \frac{r}{2}\rfloor}^{\max(k-l,l)+\lfloor\frac{r+1}{2}\rfloor} \frac{2-\delta_{i,0}}{2} \times\left(A^{k-l}_{2i-\lfloor\frac{r+1}{2}\rfloor} + A^{k-l}_{-2i-2\lfloor\frac{r}{2}\rfloor}\right)\left(A^{l}_{2i-\lfloor\frac{r+1}{2}\rfloor} + A^{l}_{-2i-2\lfloor\frac{r}{2}\rfloor}\right)q^k \tag{44}$$

$$\frac{1}{S_r^2} = 1 + \sum_{k=1}^{\infty}\sum_{l=1}^{k-1} \sum_{i=r-2\lfloor \frac{r}{2}\rfloor}^{\max(k-l,l)+\lfloor\frac{r+1}{2}\rfloor} \frac{2-\delta_{i,0}}{2} \times\left(B^{k-l}_{2i-\lfloor\frac{r+1}{2}\rfloor} - B^{k-l}_{-2i-2\lfloor\frac{r}{2}\rfloor}\right)\left(B^{l}_{2i-\lfloor\frac{r+1}{2}\rfloor} - B^{l}_{-2i-2\lfloor\frac{r}{2}\rfloor}\right)q^k \tag{45}$$

A^k_{2i} is given by equations (6) and B^k_{2i} by equations (14). For example, the normalized series expansion for $ce_4(z,q)$ up to order $\mathrm{O}(q^4)$ is

$$\begin{aligned} ce_4(z,q) = {}& \left[\cos(4z) + \left(\tfrac{1}{12}\cos(2z) - \tfrac{1}{20}\cos(6z)\right)q + \left(\tfrac{1}{192} + \tfrac{1}{960}\cos(8z)\right)q^2\right. \\ &\left. +\left(\tfrac{11}{17\,280}\cos(2z) - \tfrac{13}{96\,000}\cos(6z) - \tfrac{1}{80\,640}\cos(10z)\right)q^3 + \mathrm{O}(q^4)\right] \\ &\times\left(1 + \tfrac{17}{1800}q^2 + \tfrac{36\,283}{207\,360\,000}q^4 + \mathrm{O}(q^5)\right)^{-1/2}. \end{aligned} \tag{46}$$

Now we address the normalization of the asymptotic expansion given by equations (31). Replacing these equations in the normalization formulae (43) we eventually reach a point where we need the values of the integrals

$$\int_0^{2\pi} \frac{\mathrm{D}_m(\alpha)\,\mathrm{D}_n(\alpha)}{\sqrt{1-\frac{\alpha^2}{4\sqrt{q}}}}\,\mathrm{d}\alpha. \tag{47}$$

Sips [4] shows that for large q the values of integrals (47) can be obtained from the integrals

$$I^k_{m,n} = \int_{-\infty}^{\infty} \alpha^k\,\mathrm{D}_m(\alpha)\,\mathrm{D}_n(\alpha)\,\mathrm{d}\alpha \tag{48}$$

where k is a positive integer. We are aware that [11] does present a solution for the integrals (48), but its form is of no value for our purpose. Now we obtain solutions of integrals (48) in the form of a sum of Kronecker deltas which are suitable for our calculations. It is well known that

$$I^0_{m,n} = \sqrt{2\pi}\, m!\,\delta_{m,n}. \tag{49}$$

The generalization of the result above for $k \geqslant 0$ can be obtained from the ansatz

$$I^k_{m,n} = \sqrt{2\pi} \sum_{l=k-2\lfloor\frac{k}{2}\rfloor}^{\lfloor\frac{k+1}{2}\rfloor} \left(n!\,\tau^k_{2l-k+2\lfloor\frac{k}{2}\rfloor,n}\,\delta_{m,n-2l+k-2\lfloor\frac{k}{2}\rfloor} + m!\,\tau^k_{2l-k+2\lfloor\frac{k}{2}\rfloor,m}\,\delta_{n,m-2l+k-2\lfloor\frac{k}{2}\rfloor}\right). \tag{50}$$

Integrating (48) by parts and using the property

$$\frac{\mathrm{dD}_m(\alpha)}{\mathrm{d}\alpha} = \frac{\alpha}{2}\,\mathrm{D}_m(\alpha) - \mathrm{D}_{m+1}(\alpha) \tag{51}$$

we obtain the relation

$$I^k_{m,n} = I^{k-1}_{m+1,n} + I^{k-1}_{m,n+1} - \big(k-1\big) I^{k-2}_{m,n}. \tag{52}$$

Substituting ansatz (50) into equation (52) we obtain the following recursive formula for the coefficients $\tau^k_{m,n}$:

$$\tau^k_{m,n} = (1+\delta_{m,1})\,\tau^{k-1}_{m-1,n} + (n+1)\,\tau^{k-1}_{m+1,n+1} - (k-1)\,\tau^{k-2}_{m,n} \tag{53}$$

where $\tau^k_{k,n} = 1$ for $k > 0$, $\tau^0_{0,n} = \frac{1}{2}$, and $\tau^k_{m,n} = 0$ for $m < 0$. For example, when $k = 4$ we have

$$I^4_{m,n} = \sqrt{2\pi}\,n!\,\left[\left(3n^2+3n+\tfrac{3}{2}\right)\delta_{m,n} + (4n-2)\,\delta_{m,n-2} + \delta_{m,n-4}\right] + \sqrt{2\pi}\,m!\,\left[\left(3m^2+3m+\tfrac{3}{2}\right)\delta_{n,m} + (4m-2)\,\delta_{n,m-2} + \delta_{n,m-4}\right]. \tag{54}$$

By using equations (50) one eventually obtains the following formulae for C_r and S_r:

$$\frac{1}{C_r^2} = \frac{\sqrt{2}}{\pi q^{\frac{1}{4}}} \sum_{k=0}^{\infty} \frac{1}{(4\sqrt{q})^k} \sum_{l=0}^{k} \frac{\Gamma(k-l+\frac{1}{2})}{\Gamma(k-l+1)} \sum_{m=0}^{l}\sum_{n=0}^{k-l} \times \left[\left(1+n-2\left\lfloor\frac{n}{2}\right\rfloor\right) L_0^{(r)} + \left(1-n+2\left\lfloor\frac{n}{2}\right\rfloor\right) L_1^{(r)}\right] \tag{55}$$

$$\frac{1}{S_r^2} = \frac{\sqrt{2}}{\pi q^{\frac{1}{4}}} \sum_{k=0}^{\infty} \frac{1}{(4\sqrt{q})^k} \sum_{l=0}^{k} \frac{-\Gamma(k-l-\frac{1}{2})}{2\Gamma(k-l+1)} \sum_{m=0}^{l}\sum_{n=0}^{k-l} \times \left[\left(1-3n+6\left\lfloor\frac{n}{2}\right\rfloor\right) L_0^{(r)} + \left(1-n+2\left\lfloor\frac{n}{2}\right\rfloor\right) L_1^{(r)}\right]. \tag{56}$$

$L_0^{(r)}$ and $L_1^{(r)}$ are given by

$$L_p^{(r)} = \sum_{j=p-m}^{m} G^m_{4j-2p}\Big(|r+4j-2p|\,!\,G^{l-m}_{4j-2n-2p}\,\tau^{2k-2l}_{2n,\,r+4j-2p} + |r+4j+2n-2p|\,!\,G^{l-m}_{4j+2n-2p}\,\tau^{2k-2l}_{2n,\,r+4j+2n-2p}\Big) \tag{57}$$

where $p = 0$ or $p = 1$ and G^k_{2i} is given by equation (38).

For example, the normalization factor for $ce_r(z,q)$ up to order $\mathrm{O}(\frac{1}{q^{(5/2)}})$ is

$$\frac{1}{C_r^2} = \frac{\sqrt{2}r!}{\sqrt{\pi}q^{\frac{1}{4}}}\Bigg(1 + \frac{1}{8}\,\frac{2r+1}{\sqrt{q}} + \frac{r^4+2r^3+263r^2+262r+108}{2048\,q} + \frac{6r^5+15r^4+1280r^3+1905r^2+1778r+572}{16\,384\,q^{(3/2)}} + \frac{1}{33\,554\,432\,q^2}(5r^8+20r^7+8186r^6+24\,488r^5+1716\,437r^4 + 3392\,084r^3+5188\,652r^2+3496\,688r+1040\,576) + \mathrm{O}\left(\frac{1}{q^{(5/2)}}\right)\Bigg) \tag{58}$$

which agrees with formulae (3.12) and (3.13) of [10].

6. Conclusions

We obtain closed formulae for the generic terms of expansions of Mathieu functions of the first kind for both small and large values of q. The calculations are based on the well known ansatz for the expansions. We are able to find recursive formulae for the coefficients which, in general, depend on q and r. These formulae seem to be too complicated to eliminate the recursion except for some limiting cases.

We consider two forms of asymptotic expansion: one in terms of trigonometric functions and the other in terms of parabolic cylinder functions. In general, expansions in terms of trigonometric functions are efficient for numerical evaluation, since trigonometric functions can be evaluated in the machine co-processor. On the other hand, expansion (20) has an essential singularity at $z = \frac{\pi}{2}$. This is one motivation to address the problem of finding the generic term of asymptotic expansion in terms of parabolic cylinder functions.

We can easily obtain the generic terms of expansions of Mathieu functions of the second kind for small q from the results of this paper. The same comment applies for the derivative of Mathieu functions with respect to z. We are currently working on the problem of implementing Mathieu functions on computer algebra systems and analysing numerical methods based on the results of this paper.

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